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Elastic–plastic and limit-state analyses of frames with softening plastic-hinge models by mathematical programming

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Dedicated to Professor George Dvorak

Abstract

Frames (and more general beam systems) subjected to monotonic loading are modelled by conventional finite elements with the traditional assumption of possible plastic deformations concentrated in pre-selected "critical sections". The inelastic behaviour of these beam sections, i.e. the development of "plastic hinges", is described by piece-wise-linear constitutive models allowing for hardening and/or softening, in terms of generalized stresses and conjugate kinematic variables.

The following topics are discussed: step-by-step analysis methods, both "exact" and stepwise holonomic; path bifurcations and overall stability; limit and deformation analyses combined, as an optimization problem under complementarity constraints apt to compute the safety factor (with respect to global or local failures); numerical tests of nonconventional algorithms by means of simple representative applications.

The objective of the paper is to provide a unified methodology and to propose novel procedures for inelastic analyses of frames up to failure, in the light of recent results in mathematical programming, particularly on complementarity theory.

1. Introduction

An abundant literature nowadays provides, especially to structural engineers dealing with common structures such as frames and beam systems, limit-state and inelastic analysis methods intended to achieve a compromise between the conflicting requirements of sizmplicity and realism of results (see e.g. Massonet and Save, 1978; Kaliszky, 1989; Jiràsek and Bažant, 2002). A contribution in this direction is the objective of this paper: it presents a methodology which, in a sense, unifies direct limit and time-stepping analyses and is motivated and characterized by the preliminary surveying remarks outlined below.

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(a) The traditional idealization of "plastic hinges" is herein preserved, namely possible plastic deformations (not necessarily flexural only) are confined to "critical sections", located between adjacent conventional finite elements which model elastic beams. Softening behaviour in bending is exhibited by steel beams susceptible to flexural plastic deformations and/or local buckling (see e.g. Kim and Reid, 2001) and typically by most reinforced concrete beams (see e.g. Jiràsek and Bažant, 2002). In bending tests up to failure of over-reinforced concrete beams the moment peak is soon followed by a sloping down, softening branch in the moment-curvature plot; such branch is delayed after a more or less extended horizontal stage of relatively ductile behaviour in bending tests on under-reinforced beams (susceptible to idealizations visualized in Fig. 3). In the presence of softening, localization of inelastic flexural deformation to cross-sections (i.e. singularity of curvature giving rise to relative rotation between adjacent sections) is a fairly realistic traditional idealization, quite acceptable to structural engineering purposes (see e.g. Maier, 1968; Maier, 1971; Bažant et al., 1987; Bažant and Jiràsek, 1996; Jiràsek, 1997; Bolzon and Tin-Loi, 1999; Bažant, 2001; Jiràsek and Bažant, 2002). However, such idealization is not essential and might be relaxed in favour of "spread plasticity" models in the context of nonlocal (e.g. strain-gradient) plasticity for "regularization" (see e.g. Avery and Mahendran, 2000; Royer-Carfagni, 2001; Yang et al., 2002) by preserving essential features of most present developments.

(b) The behaviour of the critical sections (e.g. in terms of bending moment versus rotation, or in presence of moment versus axial force interaction) is described herein by a rigid-plastic or elastic–plastic hardening and/or softening models which are "piece-wise-linear" (PWL). This means that the yield functions (and plastic potentials, if distinct from them) are linear, both in the static generalized variables and in the "plastic multipliers", which act as internal variables in cases of (linear) hardening or softening. Constitutive PWL models are adopted in structural plasticity since a long time (see e.g. Maier, 1970, 1976; Capurso, 1971; Hodge, 1977; Tin-Loi, 1990; Olsen, 1998); clearly, the perfectly plastic flexural "hinge" of classical limit analysis is a very special case of PWL models. An analytical PWL representation can be given, quite accurately, to the elastic-locking behaviour implied e.g. by unilateral contact in joints and, in particular, in semirigid joints of steel frames (Corradi and Maier, 1969; Gawecki and Janinska, 1995). Also cohesive crack and interface models have been formulated in PWL format (Bolzon et al., 1995; Maier and Comi, 2000; Cocchetti et al., 2002). The PWL nature of the assumed constitutive models is essential to the present purposes, because it entails (as it will be shown in this paper) that a single mathematical construct, namely the linear complementarity problem (LCP), acquires a recurrent and central role in a variety of approaches and solution methods (even in the presence of geometric effects, provided that they are captured by geometric stiffness matrices, in the spirit of a "second order theory"). Whereas complementarity problems and their frequent involvement in elastoplasticity theory have been studied since many years (see e.g. Maier, 1970; Cohn and Maier, 1979; Lloyd Smith, 1990; Wakefield and Tin-Loi, 1990; Giambanco, 1999), some recent and promising mathematical developments considered herein turn out not to have been fully exploited so far in structural analysis, to the writers' knowledge.

(c) As a consequence of the PWL models for local nonlinear behaviour, the response of a frame structure to external actions varying proportionally (or stepwise proportionally) can be computed "exactly" (to within the approximations implied by the local models and by the overall discretization modelling) as it will be discussed herein. In the presence of softening, exact time integration may be endowed with provisions apt to capture possible branching of "equilibrium paths" and overall instability thresholds (see e.g. Bažant, 2001; Bažant and Cedolin, 1991; Biolzi and Labuz, 1993; Bolzon et al., 1997; Franchi et al., 1998; Maier et al., 1973).

(d) Under external actions monotonically increasing proportionally by a load factor (a frequent assumption in engineering practice), an elastic–plastic structure often exhibits negligible "local unloadings". Therefore, it is often practically reasonable to rule out a priori all manifestations of irreversibility by assuming path-independent constitutive models for plasticity ("holonomy", "deformation theory"), see e.g. Cohn and Maier (1979); Griffin et al. (1988); Tin-Loi and Wong (1989). With PWL models the transition from nonholonomic (i.e. path-dependent, irreversible) to holonomic descriptions of nonlinear sectional behaviours will be shown to be straightforward and to preserve the centrality of the Linear Complementarity Problem (LCP) in the analytical formulations.

(e) The above holonomy assumption permits to formulate a "combination of limit and deformation analysis" (CLDA). As shown in what follows, this means maximization of the live load amplifier under equilibrium, compatibility, constitutive models and deformation constraints, simultaneously. The resulting maximum represents the safety factor with respect to either plastic collapse or local failure or unserviceability or onset of overall instability, alternatively. Thus the limitations of classical limit analysis are overcome, at the price of the complexity of a load factor maximization under additional nonlinear, nonconvex and nonsmooth constraints. The CLDA in the above sense was originally proposed under the restrictive assumption of Drucker's constitutive stability and by use of linear programming (LP) "Simplex" method with the ad hoc "restricted basis" innovation (cf. Maier et al., 1979). Here, a CLDA is proposed which allows for softening instability and makes recourse to mathematical concepts and algorithms mostly developed in the last few years, namely "mathematical programming under equilibrium constraints" (MPEC) in nonsmooth nonconvex mechanics, see e.g. Luo et al. (1996); Mistakidis and Stavroulakis (1998); Gao et al. (2001).

Rooted in consolidated areas of structural plasticity as shown by the preceding remarks, the study expounded in this paper contains the following contributions to inelastic frame analysis: analytical, LCP-based representations of piece-wise-linearized relationships between static and kinematic generalized variables of beam cross-sections also in the presence of softening behaviour (Sections 2 and 8.1); "exact time-integration" method and step-wise and fully holonomic analyses allowing for possible bifurcations and overall instability due to softening, all resting on the unifying basis of LCP (Sections 4 and 5); combined limit and deformation analysis also in the presence of softening (Section 6); brief comments on computational aspects, with special reference to concepts and algorithms recently devised in the mathematical programming community (Section 7). A simple illustrative example is discussed in Section 8 in order to further clarify mechanical and computational consequences of softening hinges and in the light of the present results. Possible extensions of the results achieved in the paper are concisely pointed out in the conclusive Section 9.

Matrix notation is adopted throughout. Matrices and vectors are represented by bold-face characters. Transposition is indicated by superscript T. A dot marks rate, i.e. derivative with respect to ordering, not necessarily physical, time t. Vector inequalities apply componentwise.

2. Piece-wise-linear models for plastic hinge behaviour

2.1. General formulation

According to a traditional modelling in civil engineering, possible inelastic deformations in beams and frames are confined to pre-selected "critical cross-sections". Vectors $\mathbf{Q}$ and $\mathbf{q}$ will gather in what follows the sectional stress resultants (generalized variables) and, respectively, the corresponding (virtual work conjugate) kinematic variables consistent with the adopted structural model.

In the usual two-dimensional (2D) interpretations of building frames the generalized stresses gathered in vector $\mathbf{Q}$ are the bending moment and the axial force (the "interaction" between them being described by yield limit curves referred to in codes of practice); the generalized strains in vector $\mathbf{q}$ consist of relative rotation (with respect to a principal centroidal axis in the section) and axial elongation (discontinuity of displacement along centroidal axis of the beam). When the effects of axial force are negligible, as frequently in practice especially for horizontal beams, the "plastic hinge" idealization is arrived at, characterized merely by a moment–rotation relationship like those depicted in Fig. 1, including the classical special case of the "perfectly-plastic" hinge (Fig. 1a). A PWL model describing interaction between moment and axial force is schematically shown in Fig. 2.

For three-dimensional (3D) beam systems and frames, "plastic hinge" modelling may deal with axial force, two bending moments, or two bending moments and one torque and with corresponding (work conjugate) generalized strains. Illustrative special cases and the relevant detailed mathematical models are presented in Section 8.1.

A general formulation of PWL elastic–plastic models, relating in time t (generalized) strain histories $\mathbf{q}$ to (generalized) stress histories $\mathbf{Q}$ in a critical section, according to the above idealizations reads (see e.g. Maier, 1970; Cohn and Maier, 1979; Lloyd Smith, 1990):

$$\mathbf{q} = \mathbf{e} + \mathbf{p}, \quad \mathbf{e} = \mathbf{E}^{-1}\mathbf{Q}, \quad \mathbf{p} = \mathbf{V}\boldsymbol{\lambda} \tag{1}$$

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\mathbf{Q} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} = 0 \tag{2}$$

Eq. (1) split the strain vector $\mathbf{q}$ into an elastic addend $\mathbf{e}$ (unless a rigid-plastic model, $\mathbf{e} = \mathbf{0}$, is adopted) and a plastic one $\mathbf{p}$, and relate the former to the stress vector through an elastic stiffness matrix $\mathbf{E}$, the latter to the vector of plastic multipliers through matrix $\mathbf{V}$, the columns of which can be interpreted as gradients of linear (or linearized) plastic potentials (one for each, ith, yield mode, $i = 1, \ldots, y$). Vector $\boldsymbol{\varphi}$ collects the y linear, or linearized, yield functions (one for each yield mode i), the $\mathbf{Q}$-gradient of which are columns of matrix $\mathbf{N}$. Matrix $\mathbf{H}$ is the "hardening matrix". The constant vector $\mathbf{Y}$ collects "yield limits", since the ith

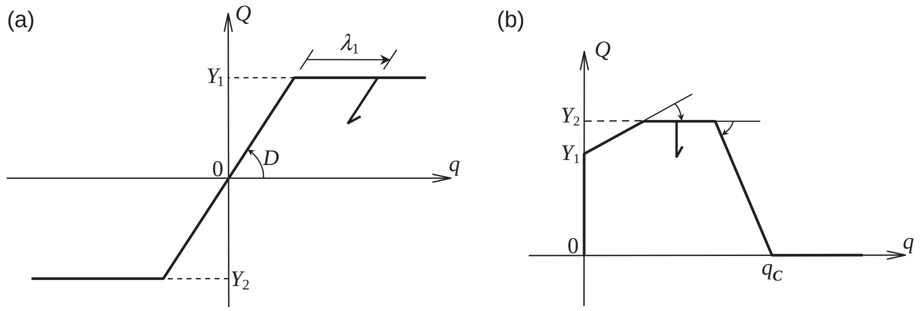

Fig. 1. PWL flexural plastic hinge models: bending moment Q versus relative rotation q. (a) Conventional perfectly plastic model; (b) a model with four yielding modes, one of them being a softening mode.

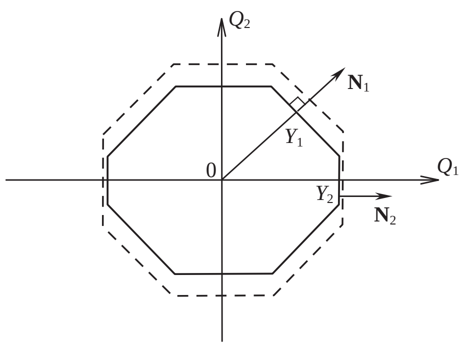

Fig. 2. PWL plastic hinge model with interaction between bending moment and axial force.

component of it represents the original distance of the ith yield plane ($i = 1, \dots, y$) from the origin in the space of generalized stresses, after a normalization which makes all columns in matrix **N** to become unit vectors in space **Q** (the same is herein assumed for matrix **V**). The y-vector $\boldsymbol{\lambda}$ gathers the "plastic multipliers". It is worth noting that in the above PWL formulation the (time-integrated) plastic multipliers $\boldsymbol{\lambda}$ acquire the role of internal variables; in fact, they govern, through Eq. (2a), the changes of the current "elastic domain", i.e. the polyhedron defined in the **Q**-space by the linear inequalities (2a). Vector $\boldsymbol{\lambda}$ represents the only memory of the past dissipation history.

The entries of the (symmetric positive-definite) elastic stiffness matrix **E** degenerate to infinite in rigid-plastic models ($\mathbf{e} = \mathbf{0}$), which are physically suitable to plastic hinges and interface models. Sometimes the entries of **E** are given fictitious values for computational convenience; in other situations they are meant to interpret the spreading of elastic deformations which accompany the inelastic ones, but, like them, are concentrated in a section as a simplifying idealization which is fairly realistic in the presence of softening behaviour.

When matrix **E** is finite, the inverse formulation of the PWL general model, governing the generalized stress response $\mathbf{Q}(t)$ to an assigned strain path $\mathbf{q}(t)$, can be easily seen to materialize in the relationships:

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\mathbf{E}\mathbf{q} - \mathbf{K}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0} \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \quad \boldsymbol{\varphi}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} = 0 \tag{3}$$

$$\mathbf{Q} = \mathbf{E}\mathbf{q} - \mathbf{E}\mathbf{V}\boldsymbol{\lambda} \tag{4}$$

having set:

$$\mathbf{K} \equiv \mathbf{H} + \mathbf{N}^{\mathrm{T}}\mathbf{E}\mathbf{V} \tag{5}$$

In both the direct and the inverse PWL model, all nonlinearities are confined to Eqs. (2) and (3), respectively. When the rate vector $\dot{\boldsymbol{\lambda}}(t)$ and, hence, $\boldsymbol{\lambda}(t)$ are computed through them, the response $\mathbf{q}(t)$ or $\mathbf{Q}(t)$ results from explicit linear transforms (1) or (4), respectively.

In terms of rates (i.e. of infinitesimal increments) starting from a state $\{\overline{\mathbf{Q}}, \overline{\boldsymbol{\lambda}}\}$ where only a subset of yield planes is active (the relevant vectors are marked by a prime), Eqs. (1) and (2) and (3)–(5) generate straightforwardly the following direct and inverse flow rules, respectively:

$$\dot{\boldsymbol{\varphi}}' = \mathbf{N}'^{\mathrm{T}}\dot{\mathbf{Q}} - \mathbf{H}'\dot{\boldsymbol{\lambda}}' \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}' \geqslant \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}'^{\mathrm{T}}\dot{\boldsymbol{\lambda}}' = 0 \tag{6}$$

$$\dot{\boldsymbol{\varphi}}' = \mathbf{N}'^{\mathrm{T}}\mathbf{E}\dot{\mathbf{q}} - \mathbf{K}'\dot{\boldsymbol{\lambda}}' \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}' \geqslant \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}'^{\mathrm{T}}\dot{\boldsymbol{\lambda}}' = 0 \tag{7}$$

2.1.1. Remarks

The following circumstances are worth noting on the preceding formulation of generalized plastic hinge models.

(a) The main peculiar feature of PWL models is the circumstance that the yield surfaces in the **Q**-space are "yield planes", which may merely translate at yielding and "interact" (in the sense that the "activation" of one can induce others to translate). Clearly, these motions and interactions are governed by vector $\boldsymbol{\lambda}$ through the hardening matrix **H**. In fact, if the gradients of yield functions $\boldsymbol{\varphi}$ (i.e. the outward normals to the yield planes) contained as columns in matrix **N** are normalized, then the current distance of the yield plane from the origin reads ($\mathbf{H}_r$ denotes the rth row of **H**):

$$Y_r + \mathbf{H}_r\boldsymbol{\lambda} = Y_r', (r = 1, \dots, y) \tag{8}$$

If matrix **H** (of order y) is diagonal, there is no interaction among yield planes ("Koiter's rule"). In classical plasticity, kinematic hardening and isotropic hardening mean rigid-body translation and, respectively, shape-preserving homothetic expansion of the entire yield locus. In the PWL context, such restrictive assumptions could easily be seen to reduce drastically (to a single one, say k and h, respectively) the available hardening/softening parameters and to imply the following specialization of matrix $\mathbf{H}$, respectively: $\mathbf{H} = k\mathbf{N}^{\mathrm{T}}\mathbf{N}$, $\mathbf{H} = h\mathbf{Y}\mathbf{v}^{\mathrm{T}}$. In the latter (isotropic) hardening rule, where $\mathbf{v}$ is an arbitrary y-vector, two special cases are worth noting: $\mathbf{v} = \{Y_1, Y_2, \dots, Y_y\}^{\mathrm{T}}$ and $\mathbf{v} = \{Y_1^{-1}, Y_2^{-1}, \dots, Y_y^{-1}\}^{\mathrm{T}}$. The former case implies the symmetry of the hardening matrix $\mathbf{H}$ and, for each yield mode i, a self-hardening modulus $H_{ii} = hY_i^2$; the latter choice generally disrupts the symmetry of $\mathbf{H}$, but confers equal self-hardening modulus for all modes.

(b) Convexity of all current "elastic domains" in the $\mathbf{Q}$-space, implied by Drucker's postulate, is obviously intrinsic in any PWL approximation. Another implication, associativity (or "normality"), means $\mathbf{N} = \mathbf{V}$ and will be accepted in what follows. In order to evidence the meaning and the instabilizing effects of the two possible violations of Drucker's postulate in PWL models, namely nonassociativity ($\mathbf{N} \neq \mathbf{V}$) and/or softening (see Maier, 1967; Palmer et al., 1967), the second order work $\delta^2\Pi$ is represented here through easy manipulations of Eq. (6), account taken of Eqs. (1):

$$\delta^2\Pi = \frac{1}{2}\delta\mathbf{Q}^{\mathrm{T}}(\delta\mathbf{q})\delta\mathbf{q} = \frac{\delta t^2}{2}[\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{E}^{-1}\dot{\mathbf{Q}} + \dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{H}'\dot{\boldsymbol{\lambda}}' + \dot{\mathbf{Q}}^{\mathrm{T}}(\mathbf{V} - \mathbf{N})\dot{\boldsymbol{\lambda}}'] \tag{9}$$

Usually plastic hinge models are rigid-plastic and, hence, the first addend on the r.h.s. in Eq. (9), i.e. the (strictly convex) elastic strain energy, vanishes ($\mathbf{E}^{-1} = \mathbf{0}$). Associativity ($\mathbf{N} = \mathbf{V}$) implies that constitutive instability ($\delta^2\Pi < 0$ for some perturbation) may occur only when $\mathbf{H}$ is nondefinite, namely with softening behaviour, which may have the crucial mechanical and computational consequences to be dealt with later herein. More specifically, when $\mathbf{N} = \mathbf{V}$ and $\mathbf{E}^{-1} = \mathbf{0}$, stability (in the sense of $\delta^2\Pi \geqslant 0$ for any $\delta\mathbf{q}$) is guaranteed by the hinge model in a given situation if and only if the hardening submatrix $\mathbf{H}'$ concerning the currently active yield modes is co-positive (i.e. if and only if $\dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{H}'\dot{\boldsymbol{\lambda}}' \geqslant 0$ for any $\dot{\boldsymbol{\lambda}}' \geqslant \mathbf{0}$).

3. Discrete formulation of frame analysis in PWL elastoplasticity

Let n, c and y denote, respectively, the numbers of: critical sections; generalized stress components in each one of them ($c = 1$ in Fig. 1; $c = 2$ in Fig. 2); yield modes considered in each of them ($y = 2, 4$ in Figs. 1a and b, respectively). A detailed discussion of some representative models is postponed to Section 8.1. Symbols $\mathbf{Q}$ and $\mathbf{q}$ (the latter with its elastic $\mathbf{e}$, if any, and plastic $\mathbf{p}$ addends) will represent henceforth the n c-vectors which gather all n c generalized stresses and strains, respectively, in the n critical sections envisaged by the overall structural model. Similarly, $\boldsymbol{\lambda}$ and $\mathbf{Y}$ will denote the n y-vector of all plastic multipliers and of all yield limits in the modelled structure, respectively. Consistently with this compact notation, matrices $\mathbf{N}$ and $\mathbf{V}$ (assumed equal in the present paper) and $\mathbf{H}$ shall contain as diagonal blocks all their local counterparts (considered in Section 2.1) formulated for all pre-selected critical cross-sections in the frame model. Thus, Eqs. (2) and (1) are recovered and re-written here, but now with associativity ($\mathbf{V} = \mathbf{N}$) and with the notational convention to analytically represent the idealized local inelastic behaviour in all the n critical sections, simultaneously:

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\mathbf{Q} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} = 0 \tag{10}$$

$$\mathbf{q} = \mathbf{N}\boldsymbol{\lambda} + \mathbf{E}^{-1}\mathbf{Q} + \mathbf{q}_t \tag{11}$$

Vector $\mathbf{q}_t$ quantifies possible deformations imposed as external actions in the critical sections; for brevity it will be assumed $\mathbf{q}_t = \mathbf{0}$ in what follows. The behaviour of beams (and of structural joints, if suitable) between critical sections is interpreted as linear elastic and is modelled in space by finite elements (FE) in terms of "natural" (i.e. intrinsic, namely unaffected by rigid body motions) generalized (nodal) variables. Covering the whole set of beam FEs before their assemblage and marking by stars the relevant comprehensive symbols we can write:

$$\mathbf{Q}^* = \mathbf{E}^*\mathbf{e}^*, \quad \mathbf{e}^* = \mathbf{q}^* - \mathbf{q}^*_t \tag{12}$$

In Eq. (12b) the imposed generalized strains $\mathbf{q}^*_t$, such as those due to thermal effects, will be ignored for brevity in the sequel ($\mathbf{q}^*_t = \mathbf{0}$).

Let us denote by $\mathbf{u}$ the vector of all d.o.f. of the overall assembled FE model, account taken of the ground-constraints supposed to be fixed. Marking by caps the symbols concerning both the inelastic critical sections and the elastic FE, geometric compatibility and equilibrium of the structural model are expressed by the following equations, respectively:

$$\hat{\mathbf{q}} = \begin{Bmatrix} \mathbf{q} \\ \mathbf{q}^* \end{Bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}^* \end{bmatrix} \mathbf{u} = \hat{\mathbf{C}}\mathbf{u} \tag{13}$$

$$\begin{bmatrix} \mathbf{C}^{\mathrm{T}} & \mathbf{C}^{*^{\mathrm{T}}} \end{bmatrix} \begin{Bmatrix} \mathbf{Q} \\ \mathbf{Q}^* \end{Bmatrix} = \hat{\mathbf{C}}^{\mathrm{T}}\hat{\mathbf{Q}} = \mathbf{F} \tag{14}$$

In Eq. (14) $\mathbf{F}(t)$ represents the time-history input vector of nodal forces "equivalent" (in the sense of the adopted FE modelling) to given loads acting on beams and on joints. As usual in practice, the unloaded ($\mathbf{F} = \mathbf{0}$) and unstressed ($\hat{\mathbf{Q}} = \mathbf{0}$) situation is assumed as initial condition.

Eqs. (10)–(14) (referred to henceforth as formulation A) constitute the relation set which governs the elastic–plastic response to proportional loading of the FE-modelled frames with PWL models of the plastic deformability confined to the n critical sections.

Two alternative, more compact formulations (B and C) of the above structural analysis problem are derived below by trivial algebraic manipulations.

Let Eqs. (11) and (12a) be used to substitute generalized stresses $\hat{\mathbf{Q}}$ in Eqs. (10) and (14) and the compatibility Eq. (13) to substitute generalized strains $\hat{\mathbf{q}}$; let matrix $\hat{\mathbf{E}}$ gather as its diagonal blocks the block-diagonal elastic stiffness matrices $\mathbf{E}$ and $\mathbf{E}^*$ of the not yet assembled constituents after FE modelling. Thus, the governing set of relations becomes (formulation B):

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\mathbf{E}\mathbf{C}\mathbf{u} - \mathbf{A}\boldsymbol{\lambda} - \mathbf{Y} \tag{15}$$

$$\hat{\mathbf{S}}\mathbf{u} - \mathbf{C}^{\mathrm{T}}\mathbf{E}\mathbf{N}\boldsymbol{\lambda} = \mathbf{F} \tag{16}$$

$$\boldsymbol{\varphi} \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} = 0 \tag{17}$$

having set:

$$\mathbf{A} \equiv \mathbf{H} + \mathbf{N}^{\mathrm{T}}\mathbf{E}\mathbf{N}, \quad \hat{\mathbf{S}} \equiv \hat{\mathbf{C}}^{\mathrm{T}}\hat{\mathbf{E}}\hat{\mathbf{C}} \tag{18}$$

Since the elastic stiffness matrix $\hat{\mathbf{S}}$ of the whole assembled and constrained FE model, is positive definite (and symmetric), the displacements $\mathbf{u}$ can be substituted from Eq. (15) through Eq. (16). This leads to the more compact formulation C:

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\mathbf{Q}^{\mathrm{e}} - \mathbf{B}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} = 0 \tag{19}$$

where

$$\mathbf{Q}^{\mathrm{e}} \equiv \mathbf{E}\mathbf{C}\hat{\mathbf{S}}^{-1}\mathbf{F} \tag{20}$$

$$\mathbf{B} = \mathbf{H} - \mathbf{N}^{\mathrm{T}}\mathbf{Z}\mathbf{N}, \quad \mathbf{Z} \equiv \mathbf{E}\mathbf{C}\hat{\mathbf{S}}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{E} - \mathbf{E} \tag{21}$$

It is worth noting that formulation C can be arrived at from A by a more direct mechanical argument: in Eq. (10a) the vector $\mathbf{Q}$ of the actual generalized stresses in the n critical sections can be conceived as the sum of the elastic stress response $\mathbf{Q}^{\mathrm{e}}$ to the given load $\mathbf{F}(t)$ and of the elastic self-equilibrated response $\mathbf{Q}^{\mathrm{s}} = \mathbf{Z}\mathbf{N}\boldsymbol{\lambda}$

to the unknown plastic strains conceived as imposed strains, $\mathbf{Z}$ being an influence matrix of generalized self-stresses in the critical sections due to generalized strains imposed there. It is easily recognized that the elastic stress vector $\mathbf{Q}^e$ identifies with the vector defined by Eq. (20); matrix $-\mathbf{Z}$, Eq. (21b), turns out to be symmetric, positive semidefinite.

4. "Exact" elastoplastic analysis, allowing for instability and bifurcation thresholds

4.1. Step-by-step method for "exact" solutions

In view of its mathematical peculiarities consequent to the adopted PWL constitutive models for the selected critical sections, the nonlinear initial-value problem formulated in Section 3 can be numerically solved by the procedure outlined in what follows.

The compact formulation C, Eqs. (19)–(21) will be considered below, but all considerations can be straightforwardly transferred to formulations A or B.

The external actions will be conceived henceforth as the sum of dead loads $\mathbf{F}_\mathrm{D}$ generating a purely elastic response $\mathbf{Q}_\mathrm{D}^e$ at $t = 0$ (so that $\boldsymbol{\lambda} = \mathbf{0}$ at $t = 0$) and of live loads $\mathbf{F}_\mathrm{L}$ with consequent stresses $\mathbf{Q}_\mathrm{L}^e$, both proportionally amplified in time t by load factor $\mu \geqslant 0$:

$$\mathbf{F} = \mathbf{F}_\mathrm{D} + \mu\mathbf{F}_\mathrm{L}, \quad \mathbf{Q}^e = \mathbf{Q}_\mathrm{D}^e + \mu\mathbf{Q}_\mathrm{L}^e \tag{22}$$

The time-independent ("inviscid") property implied by the assumed elastic–plastic constitutive models for the frame makes time t an ordinal, chronological variable (not necessarily the physical time) and permits to identify t with the load factor μ (and hence to set $\dot{\mu} = 1$), as long as this increases.

Barred symbols will denote quantities which have already been computed, at the beginning $\bar{t}$ of the current nth step. Single and double primes will mark the subvectors in $\boldsymbol{\varphi}$ and $\boldsymbol{\lambda}$ (and consequently the submatrices in $\mathbf{N}$, $\mathbf{H}$ and $\mathbf{B}$, Eq. (21)) which correspond to yield modes currently active ($\overline{\boldsymbol{\varphi}'} = \mathbf{0}$) and non active ($\overline{\boldsymbol{\varphi}''} < \mathbf{0}$), respectively.

A peculiar strategy, called herein "exact", for solutions of the nonlinear initial-value problem (19)–(21), can be outlined as follows.

(a) At the known state $\{\bar{\mu}_n, \bar{\boldsymbol{\lambda}}_n\}$, solve the following LCP in rates for given rate $\dot{\mathbf{Q}}_\mathrm{L}^e = \dot{\mu}\mathbf{Q}_\mathrm{L}^e$ with $\dot{\mu} = 1$, and, if no solution exists, solve it for $\dot{\mu} = -1$:

$$\dot{\boldsymbol{\varphi}}' = \mathbf{N}'^\mathrm{T}\mathbf{Q}_\mathrm{L}^e\dot{\mu} - \mathbf{B}'\dot{\boldsymbol{\lambda}}' \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}' \geqslant \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}'^\mathrm{T}\dot{\boldsymbol{\lambda}}' = 0 \tag{23}$$

Let $\overline{\dot{\boldsymbol{\lambda}}'}_n$ be the (or a) solution of this first problem in rates, account taken that $\overline{\dot{\boldsymbol{\lambda}}''}_n = \mathbf{0}$.

(b) Noting that the LCP of Eq. (23) is linearly homogeneous in an amplifier Δt of both data and variables, compute the value $\Delta\bar{\mu}_n$ such that some new yield plane is "activated", i.e. is reached at the load factor $\bar{\mu}_n + \Delta\bar{\mu}_n$.

$$\Delta t_n = \max_{\Delta t}\{\Delta t | \overline{\boldsymbol{\varphi}}''_n + (\mathbf{N}''^\mathrm{T}\mathbf{Q}_\mathrm{L}^e\dot{\mu} - {}''\mathbf{B}'\overline{\dot{\boldsymbol{\lambda}}'}_n)\Delta t \leqslant \mathbf{0}\}, \quad \Delta\bar{\mu}_n = \dot{\mu}\Delta t_n \tag{24}$$

with self-evident meaning of the symbols; in particular: ${}''\mathbf{B}'$ is the submatrix of $\mathbf{B}$, Eqs. (21a), formed by the intersection of its rows corresponding to $\boldsymbol{\varphi}'' < \mathbf{0}$, with its columns corresponding to $\boldsymbol{\varphi}' = \mathbf{0}$. Where the LCP in rates (23) has no solution for $\dot{\mu} = 1$, a negative $\Delta\bar{\mu}_n$ characterizes a post-peak behaviour or, as an alternative, elastic unloading.

(c) In view of the new active and inactive yield modes, re-arrange vectors and matrices by re-decomposition into primed and double primed portions. Thereafter go to (a) and repeat the operative sequence (a) and (b) starting from $\bar{\mu}_{n+1} = \bar{\mu}_n + \Delta\bar{\mu}_n$, $\bar{\boldsymbol{\lambda}}_{n+1} = \bar{\boldsymbol{\lambda}}_n + \bar{\dot{\boldsymbol{\lambda}}}_n\Delta t_n$.

(d) Defining μ_M as the top load factor at the end of the proportional loading, if it is reached (it may be unreachable), stop when $\Delta\overline{\mu}_1 + \Delta\overline{\mu}_2 + \cdots + \Delta\overline{\mu}_n + \cdots + \Delta\overline{\mu}_l \geqslant \mu_M$ and, in the case of strict inequality, reduce the last substep $\Delta\overline{\mu}_l$, so that the equality holds.

Summing up the results of steps, the history of plastic multipliers and, through Eq. (11) with $\mathbf{E}^{-1} = \mathbf{0}$, of deformations in the critical sections due to the proportional loading $\mu(t)$ can be reconstructed (reordering of components is tacitly assumed for vectors $\boldsymbol{\lambda}$):

$$\Delta\boldsymbol{\lambda}(\mu) = \left\{ \begin{array}{c} \overline{\dot{\boldsymbol{\lambda}}'}_1 \\ \overline{\dot{\boldsymbol{\lambda}}''}_1 = \mathbf{0} \end{array} \right\} \Delta t_1 + \left\{ \begin{array}{c} \overline{\dot{\boldsymbol{\lambda}}'}_2 \\ \overline{\dot{\boldsymbol{\lambda}}''}_2 = \mathbf{0} \end{array} \right\} \Delta t_2 + \cdots \tag{25}$$

Vector $\overline{\boldsymbol{\lambda}}(\mu)$ defines through Eq. (16) the nodal displacement response $\mathbf{u}(\mu)$ and through (13) the strain history, $\hat{\mathbf{q}}(\mu)$, whence the generalized stress path $\mathbf{Q}^*(\mu)$ in beams follows through Eq. (12) and that in critical sections through $\mathbf{Q}^S = \mathbf{Z}\mathbf{N}\boldsymbol{\lambda}$.

4.2. Remarks

(a) The denomination "exact" for the above time-integration procedure is intended to emphasize the fact that it does not imply any approximation (besides, of course, round-off errors) additional to the ones implicit in the PWL constitutive models at critical sections. On the contrary, customary procedures do imply additional errors and are susceptible to instability even in the absence of softening (like for the "explicit" time integration) in the sense of lack of disturbance contractivity along the step sequence, see e.g. Feijóo and Zouain (1988); Simo and Govindjee (1991); Comi et al. (1992); Kulkarni et al. (1995); Belytschko et al. (2000); Zienkiewicz and Taylor (2000). Stability in this sense might be proven for the present "exact" strategy even in the presence of softening.

(b) The LCP in rates Eq. (23) is fully equivalent to the following quadratic programming (QP) problem:

$$\min_{\dot{\boldsymbol{\lambda}}'}\{\dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{B}'\dot{\boldsymbol{\lambda}}' - \dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{N}'^{\mathrm{T}}\mathbf{Q}^{\mathrm{e}}_{\mathrm{L}}\dot{\mu}\} = 0 \tag{26}$$

$$\text{subject to:}\quad \mathbf{N}'^{\mathrm{T}}\mathbf{Q}^{\mathrm{e}}_{\mathrm{L}}\dot{\mu} - \mathbf{B}'\dot{\boldsymbol{\lambda}}' \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}' \geqslant \mathbf{0} \tag{27}$$

where min $= 0$ specifies that the minimization leads to a solution of the original LCP rate problem if, and only if, the achieved minimum is zero.

The equivalence between problems (23) and (26)–(27) is almost self-evident. In fact, over the feasible domain defined by the linear inequalities (23a,b), the scalar product $-\dot{\boldsymbol{\varphi}}'^{\mathrm{T}}\dot{\boldsymbol{\lambda}}'$ is nonnegative. This product becomes the objective function (26) through the Eq. (23a) and vanishes for some vector $\dot{\boldsymbol{\lambda}}'$ by reaching its minimum under the constraints (27) if, and only if, vector $\dot{\boldsymbol{\lambda}}'$ solves the LCP (23).

Usually, i.e. for general matrices $\mathbf{H}$ and, hence, $\mathbf{B}$, the QP formulated by Eqs. (26) and (27) is not convex, since its objective function may be nonconvex. The present one, is a particular case of nonconvex optimization, a growing field in mathematics and mechanics, see e.g. Gao (1999); Mistakidis and Stavroulakis (1998).

(c) If applied to formulation A or B of the time-stepping PWL elastic–plastic analysis of frames, the central repetitive role of LCP is preserved. In fact, the presence of free (sign unconstrained) variables and linear equations does not alter the essential mathematical features, nor the solution processes. The number of unknowns obviously increases, but, as a partial compensation, the inversion of the overall stiffness matrix $\mathbf{S}$ is avoided since the influence matrix $\mathbf{Z}$ no longer needs be generated.

(d) Suppose that Drucker's postulate holds and, hence, softening is ruled out. Then the hardening matrix $\mathbf{H}$ is positive semidefinite, and so is $\mathbf{B}$, since so is $-\mathbf{Z}$ by its very meaning and origin (in fact, the elastic strain energy stored in the frame if generalized strains $\mathbf{p}$ were in the critical sections would amount to

$-\mathbf{p}^{\mathrm{T}}\mathbf{Z}\mathbf{p}/2$). In this case the LCP, Eq. (23), can be interpreted as the Kuhn–Tucker optimality conditions of a convex QP problem, fully equivalent to it. From this QP, its (still convex) dual QP can be generated through well-consolidated mathematical programming concepts, see e.g. Cottle et al. (1992). Similar statements hold for formulation B. This remark leads, for the special discrete frame models in point, to the two pairs of dual extremum theorems which characterize by the linearly constrained minimum of a convex quadratic functional the rate solution in Druckerian plasticity. One pair concerns the rate version of formulation B, Eqs. (15)–(18), the other pair the condensed rate formulation, Eqs. (19)–(21), see e.g. Capurso and Maier (1970). These two pairs are computer-oriented supplements to the pair of classical theorems due to Greenberg and Prager–Hodge which imply unconstrained and convex but nonsmooth minimization. It is worth noting that the above pairs of theorems in rates hold throughout the domain of plasticity stable in Drucker's sense, independently from PWL assumptions, see e.g. Lubliner (1990).

(e) As long as Drucker stability postulate is fulfilled at the constitutive level, the LCP in rates, Eq. (23), to be solved at each step, is known to exhibit the following main properties: (i) in hardening plasticity there always exists a single solution; (ii) in perfect plasticity ($\mathbf{H} = \mathbf{0}$) there is either one solution or an infinite number of solutions. In the latter case (ii) the solution multiplicity forms either a bounded set, which mechanically means "pseudo-mechanism" (i.e. an infinity of bounded equilibrium configurations under the same loads) or an unbounded one (a "feasible ray") which represents a collapse mechanism in the sense of limit analysis and characterizes the current live load multiplier μ as the safety factor. These properties can be regarded as mechanical interpretations of consolidated mathematical features of LCP related to the nature of its matrix, see e.g. Cottle et al. (1992).

In fact, matrix $\mathbf{H}$ and, hence, matrices $\mathbf{A}$, Eq. (18a), and $\mathbf{B}$, Eq. (21a), are positive-definite in the former case (i) of hardening, positive-semidefinite in the latter case (ii) of perfect plasticity. Clearly, as a consequence, their diagonal-block submatrices involved in the current LCP in rates, like Eq. (23), are positive definite in case (i) and thus guarantee the existence and uniqueness of the rate solution; they may be positive semidefinite in case (ii) and, hence, such guarantee fails.

4.3. On path branching and overall instability

In the presence of constitutive instability due to softening in the plastic hinges, the matrices $\mathbf{H}$ (and, hence, $\mathbf{A}$ and $\mathbf{B}$, Eqs. and (18a) and (21a), and their current submatrices as a consequence) are not necessarily positive-semidefinite: they may be indefinite. Therefore, the LCP in rates, Eqs. (23), may have a discrete or continuum multiplicity of solutions or no solution, see e.g. Cottle et al. (1992).

Mechanically, lack of solution to LCP (23) for $\dot{\mu} = 1$ characterizes a load peak. This peak, say $\breve{\mu}$, of the load factor may be a local maximum, i.e. followed by a decrease and subsequent growth (e.g. due to a nonsoftening stage following the softening one in plastic hinges). Alternatively, the load peak may be "global", namely the load factor $\breve{\mu}$ represents the safety factor s with respect to the exhaustion of the carrying capacity of the structure, generally lower than the hypothetical one with respect to plastic collapse in perfect plasticity obtained by removing softening (i.e. by setting $\mathbf{H} = \mathbf{0}$) in the hinge models.

A discrete multiplicity of (nonproportional) solutions means "bifurcation", i.e. branching of the equilibrium path followed by the structure along the assigned loading history, see e.g. Biolzi and Labuz (1993); Franchi et al. (1998). The branch that "nature will choose" can be selected by thermodynamic criteria, see e.g. Bažant and Cedolin (1991). For the present category of structures, the second-order work can be expressed as follows (to within the factor $\delta t^2/2$):

$$\dot{\mathbf{F}}_{\mathrm{L}}^{\mathrm{T}}\dot{\mathbf{u}} = \dot{\mathbf{Q}}^{*\mathrm{T}}\dot{\mathbf{q}}^{*} + \dot{\mathbf{Q}}^{\mathrm{T}}\dot{\mathbf{q}} = \dot{\mathbf{u}}^{\mathrm{T}}\mathbf{S}^{*}\dot{\mathbf{u}} + \dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{E}^{-1}\dot{\mathbf{Q}} + \dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{H}'\dot{\boldsymbol{\lambda}}' \tag{28}$$

where $\dot{\mathbf{F}}_{\mathrm{L}}$ is the given vector of live load rates (here $\dot{\mathbf{F}}_{\mathrm{L}} = \dot{\mu}\mathbf{F}_{\mathrm{L}}$) and $\dot{\mathbf{u}}$ the displacement rate response to it.

On the r.h.s. (internal work) of the virtual work equation (28a) the former addend (starred symbols) concerns the elastic beams and is expressed in (28b) as a quadratic form associated to the relevant submatrix $\mathbf{S}^* = \mathbf{C}^{*\mathrm{T}}\mathbf{E}^*\mathbf{C}^*$ of the stiffness matrix $\hat{\mathbf{S}}$, Eq. (18b); the latter addend concerns plastic hinges active in the infinitesimal loading step $\dot{\mathbf{F}}_{\mathrm{L}}\delta t$ (starting from the known current situation $\overline{\mu}$, $\overline{\boldsymbol{\lambda}}$, $\overline{\mathbf{Q}}$) and is expressed in Eq. (28b) according to Eq. (9), with $\mathbf{V} = \mathbf{N}$ since normality was assumed in the present context. The usual denial of elasticity in the plastic hinge models ($\mathbf{E}^{-1} = \mathbf{0}$) removes the second of the three addends in Eq. (28). Vectors $\dot{\mathbf{Q}}$ and $\dot{\boldsymbol{\lambda}}$ are related to each other as parts of the solution to the LCP (23) in the overall "exact" analysis for given $\dot{\mu}$. Clearly, if various rate solutions within the multiplicity share the maximum of work (28), then only imperfections decide which one would materialize in nature. Formally similar considerations apply to quasi-brittle fracture simulated by cohesive crack models (Bolzon et al., 1997).

The customary stability criterion under conservative load can be stated as follows (see e.g. Bažant and Cedolin, 1991): a system is (strictly) stable if and only if, for all infinitesimal admissible kinematic disturbances, the changes of external actions needed to preserve equilibrium perform positive (second order) work.

In order to apply this notion to the class of the frame models in point endowed with the n critical sections modelled as rigid-plastic ($\mathbf{E}^{-1} = \mathbf{0}$) and associative ($\mathbf{N} = \mathbf{V}$), we can use Eq. (28) to express the global second-order internal work $\delta^2\overline{\Pi}$, which is now understood as a function of the virtual disturbance displacement vector $\dot{\mathbf{u}}\delta t$:

$$2\delta^2\overline{\Pi} = \delta\hat{\mathbf{Q}}^{\mathrm{T}}(\delta\mathbf{u})\delta\hat{\mathbf{q}}(\delta\mathbf{u}) = (\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{S}^*\dot{\mathbf{u}})\delta t^2 + \dot{\boldsymbol{\lambda}}'^{\mathrm{T}}(\dot{\mathbf{u}})\mathbf{H}'\dot{\boldsymbol{\lambda}}'(\dot{\mathbf{u}})\delta t^2 \tag{29}$$

An alternative, more useful expression of $\delta^2\overline{\Pi}$ can be obtained by the following path of reasoning (formally similar to the one adopted for fracture analysis in Cen and Maier (1992)). The generic virtual configuration change $\dot{\mathbf{u}}$ is conceived as generated in two stages: (a) the plastic deformations $\dot{\mathbf{p}} = \mathbf{N}\dot{\boldsymbol{\lambda}}$ are imposed in the critical sections with the consequent configuration change $\dot{\mathbf{u}}^{\mathrm{p}}$ and stress increments $\dot{\hat{\mathbf{Q}}}^{\mathrm{p}}$; (b) a "complementary", purely elastic process $\dot{\mathbf{u}}^c = \dot{\mathbf{u}} - \dot{\mathbf{u}}^{\mathrm{p}}$, endowed, by its definition, with the kinematic property $\mathbf{C}(\dot{\mathbf{u}} - \dot{\mathbf{u}}^{\mathrm{p}}) = \dot{\mathbf{q}}^c = \mathbf{0}$ in the hinges.

Since stresses $\dot{\hat{\mathbf{Q}}}^{\mathrm{p}}$ generated at stage (a) are self-equilibrated and beam strains $\dot{\mathbf{q}}^{*c} = \dot{\mathbf{e}}^{*c}$ at stage (b) are compatible, i.e. $\dot{\mathbf{q}}^{*c} = \mathbf{C}(\dot{\mathbf{u}} - \dot{\mathbf{u}}^{\mathrm{p}})$, the mutual (or "indirect") work, $\dot{\hat{\mathbf{Q}}}^{pT}\dot{\mathbf{q}}^{*c} = 0$, of the two stages is zero by the virtual work principle. Therefore, the total work at the end of the two stages amounts to:

$$2\delta^2\overline{\Pi} = (\dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{H}'\dot{\boldsymbol{\lambda}}' - \dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{N}'^{\mathrm{T}}\mathbf{Z}\mathbf{N}'\dot{\boldsymbol{\lambda}}' + \dot{\mathbf{e}}^{*c\mathrm{T}}\mathbf{E}^*\dot{\mathbf{e}}^{*c})\delta t^2 \tag{30}$$

In fact, the work performed in (a) includes the first addend in (30) representing the contribution of the hinges, and the second addend corresponding to the total elastic energy stored in the beams; the third addend is additional elastic energy in the beams due to stage (b). This stage is optional, in the sense that the hypothetical external agency which checks overall stability of the frame may operate through $\dot{\mathbf{p}}'$ only. A kinematic disturbance with $\dot{\mathbf{u}}^c = \mathbf{0}$ belongs to the set of virtual disturbances and exhibits minimum second order work for equal disturbances governed by $\dot{\mathbf{p}}'$ in active plastic hinges.

As a consequence of Eq. (30) and of the preceding remarks on it, the following operative criteria can be stated for overall stability.

The considered frame is stable in a known situation of its loading history if, and only if, matrix $\mathbf{B}'$ is copositive, namely iff:

$$\dot{\boldsymbol{\lambda}}'^{\mathrm{T}}\mathbf{B}'\dot{\boldsymbol{\lambda}}' = \dot{\boldsymbol{\lambda}}'^{\mathrm{T}}(\mathbf{H}' - \mathbf{N}'^{\mathrm{T}}\mathbf{Z}\mathbf{N}')\dot{\boldsymbol{\lambda}}' \geqslant 0 \quad \forall\dot{\boldsymbol{\lambda}}' \geqslant \mathbf{0} \tag{31}$$

The copositiveness of the current hardening matrix $\mathbf{H}'$ turns out to be sufficient, not necessary, for overall stability. Of course, the traditional distinction between loose and strict stability (the latter characterized by strict inequalities) may be computationally useful in the present context as well.

5. Stepwise holonomic and fully holonomic analysis

The exact integration which precedes (Section 4.1) exhibits the peculiar feature that the number of substeps $\Delta\mu_i$ $(i = 1, 2, \ldots, l)$ within each given loading interval $0 \leqslant \mu \leqslant \mu_M$ cannot be chosen a priori, but depends on the adopted model and on the given loading path. Clearly, this fact becomes computationally disadvantageous when numerous yield modes can be activated in many critical sections simultaneously and at small intervals.

In order to avoid possibly drastic reduction of substep amplitudes $\Delta\mu_i$, let us assume that the path-dependence does not hold within each step; in other terms, the intrinsic irreversibility of the plastic hinge model is accounted for only in updating the internal variables at the transition from step to step. Such "stepwise holonomic" interpretation of the evolution of a dissipative system (an interpretation at the basis of practically all time-integration schemes in computational plasticity), if applied to the PWL in point, leads to a step-governing relationship once again formulated in a LCP format (Maier, 1970; De Donato and Maier, 1972; Franchi and Genna, 1991).

Starting from a state characterized by a known vector $\{\overline{\mu}_n, \overline{\boldsymbol{\lambda}}_n\}$ (barred symbols for known variables), consider the LCP concerning the given finite increment $\Delta\mu_n$ and the unknowns $\boldsymbol{\varphi}_{n+1}, \Delta\boldsymbol{\lambda}_n$:

$$\boldsymbol{\varphi}_{n+1} = \overline{\boldsymbol{\varphi}}_n + \Delta\boldsymbol{\varphi}_n = \overline{\boldsymbol{\varphi}}_n + \mathbf{N}^{\mathrm{T}}\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\Delta\mu_n - \mathbf{B}\Delta\boldsymbol{\lambda}_n \leqslant \mathbf{0}, \quad \Delta\boldsymbol{\lambda}_n \geqslant \mathbf{0} \tag{32}$$

$$\boldsymbol{\varphi}_{n+1}^{\mathrm{T}}\Delta\boldsymbol{\lambda}_n = 0 \tag{33}$$

where

$$\overline{\boldsymbol{\varphi}}_n = \mathbf{N}^{\mathrm{T}}\mathbf{Q}_{\mathrm{D}}^{\mathrm{e}} + \mathbf{N}^{\mathrm{T}}\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\overline{\mu}_n - \mathbf{B}\overline{\boldsymbol{\lambda}}_n - \mathbf{Y} \tag{34}$$

denoting by $\mathbf{Q}_{\mathrm{D}}^{\mathrm{e}}$ and $\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}$ the vectors of all generalized stresses in all critical sections due to dead and reference live loads, respectively, in a hypothetically linear-elastic structural response to them.

Note that now, in principle, all yield modes contained in the frame model are involved, not only the active ones (therefore primes on symbols of Section 4 disappear).

Full equivalence can be noted between LCP (32)–(33) and the following generally nonconvex QP problem:

$$\min_{\Delta\boldsymbol{\lambda}_n}\{-\overline{\boldsymbol{\varphi}}_n^{\mathrm{T}}\Delta\boldsymbol{\lambda}_n - \Delta\boldsymbol{\lambda}_n^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\Delta\mu_n + \Delta\boldsymbol{\lambda}_n^{\mathrm{T}}\mathbf{B}\Delta\boldsymbol{\lambda}_n\} = 0 \tag{35}$$

$$\text{subject to:} \quad \overline{\boldsymbol{\varphi}}_n + \mathbf{N}^{\mathrm{T}}\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\Delta\overline{\mu}_n - \mathbf{B}\Delta\boldsymbol{\lambda}_n \leqslant \mathbf{0}, \quad \Delta\boldsymbol{\lambda}_n \geqslant \mathbf{0} \tag{36}$$

Clearly, if the increments are regarded as infinitesimal ($\Delta\overline{\mu}_n \to \dot{\mu}\delta t$), the LCP (32)–(33) formally reduces to its counterpart in rates Eq. (23) and the QP (35)–(36) to Eqs. (26) and (27).

When the incremental plastic multiplier vector $\Delta\boldsymbol{\lambda}_n$ has been obtained by solving the LCP (32)–(33) or the equivalent QP problem, the following closed-form expressions, derived from Eqs. (11), (13) and (16), provide the increments of other variables (generalized strains and stresses in the critical sections and in the beam FEs and the nodal displacements throughout the frame model):

$$\mathbf{p}_{n+1} = \mathbf{N}\boldsymbol{\lambda}_{n+1}, \quad \mathbf{u}_{n+1} = \hat{\mathbf{S}}^{-1}(\mathbf{F}_{\mathrm{D}} + \mathbf{F}_{\mathrm{L}}\overline{\mu}_{n+1}) + \hat{\mathbf{S}}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{E}\mathbf{N}\boldsymbol{\lambda}_{n+1} \tag{37}$$

$$\hat{\mathbf{q}}_{n+1} = \hat{\mathbf{C}}\mathbf{u}_{n+1}, \quad \mathbf{Q}_{n+1}^{*} = \mathbf{E}^{*}\mathbf{e}_{n+1}^{*}, \quad \mathbf{Q}_{n+1} = \mathbf{Q}_{\mathrm{D}}^{\mathrm{e}} + \mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\overline{\mu}_{n+1} + \mathbf{Z}\mathbf{p}_{n+1} \tag{38}$$

Let us consider now as starting situation the original unloaded state, which implies $\overline{\boldsymbol{\varphi}}_n = \overline{\boldsymbol{\varphi}}_0 < \mathbf{0}$, $\boldsymbol{\lambda}_n = \mathbf{0}$, $\overline{\mu}_n = 0$. Then Eqs. (32)–(33) are specialized to the following LCP formulation of the single-step, fully holonomic (path-independent, reversible, nonlinear elastic) analysis of the modelled frame under given live load factor $\overline{\mu}$:

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}(\mathbf{Q}_{\mathrm{D}}^{\mathrm{e}} + \mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\overline{\mu}) - \mathbf{B}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0}, \quad \boldsymbol{\lambda} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda} = 0 \tag{39}$$

Obviously, an equivalent QP formulation results from a similar specialization of Eqs. (35)–(36).

The meaning and implications of the above formulations in finite increments are clarified by the following remarks.

(a) The response of an elastic–plastic structure to a loading process over a time interval, say $[t_n, t_{n+1}]$, is said to exhibit ‘‘regularly progressive yielding’’ (RPY) when every yield mode r, if it is active at an instant in the interval, is not subsequently unstressed (i.e. is not subjected to ‘‘local unloading’’ $\dot{\varphi}_r < 0$). Formally, with the present notation, RPY means that if $\dot{\lambda}_r(t^*) > 0$ (and, hence, $\dot{\varphi}_r = 0$) at $t^*(t_n \leqslant t^* \leqslant t_{n+1})$, then $\varphi_r(t) = 0$ at any $t \geqslant t^*$, with $t_n \leqslant t^* \leqslant t \leqslant t_{n+1}$, $r = 1, \ldots, \overline{y}$ ($\overline{y}$ being the number of all yield modes envisaged in all critical sections in the frame model).

It is easily seen that if RPY in the above sense holds for a finite increment process under $\Delta\mu_n$, then the complementarity relation Eq. (33) is fulfilled for that process. Clearly, the converse is not necessarily true. Therefore, in the case of RPY the solution (or solutions) to the LCP (32)–(33) turns out to be ‘‘exact’’ (i.e. coincides with that provided by the procedure devised in Section 4). If RPY is only assumed but does not occur (i.e. some yield modes are unstressed), Eqs. (32) and (33) represent an approximate interpretation of the actual step process. Such interpretation is called here ‘‘stepwise holonomic’’, i.e. reversible (or path-independent) in the step $\Delta t_n = t_{n+1} - t_n$, because Eqs. (32) and (33) turn out to be verified (i.e. obtained by integrations over Δt_n) even if $\dot{\lambda}_r$ is not sign-constrained but is still complementary to φ_r at any time t during the step.

(b) Now let the further hypothesis of ‘‘no new yielding’’ over Δt_n be assumed for the (undefined) actual path over Δt_n, formally: if $\varphi_{rn} < 0$ (at t_n), then $\dot{\lambda}_r(t) = 0$ at any instant t, with $t_n \leqslant t \leqslant t_{n+1}$ for all yield modes $r = 1, \ldots, y$. Then $\boldsymbol{\varphi}_n^{\mathrm{T}}\Delta\boldsymbol{\lambda}_n = 0$, and, hence, Eqs. (32) and (33) generate the following LCP in finite increments only, involving only the yield modes which are active ($\varphi_{rn} = 0$) at the onset of the loading step (this reduction of the problem size is pointed out again by means of primes on the relevant symbols):

$$\Delta\boldsymbol{\varphi}_n' = \mathbf{N}'^{\mathrm{T}}\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\Delta\mu_n - \mathbf{B}'\Delta\boldsymbol{\lambda}_n' \leqslant \mathbf{0}, \quad \Delta\boldsymbol{\lambda}_n' \geqslant \mathbf{0}, \quad \Delta\boldsymbol{\varphi}_n'^{\mathrm{T}}\Delta\boldsymbol{\lambda}_n' = 0 \tag{40}$$

(c) In computational plasticity time-stepping analyses of structures are generally stepwise holonomic, carried out by finite difference schemes resting on the following assumptions: the constitutive law is enforced only at some instant $t_\rho = t_n + \rho\Delta t_n$, $0 \leqslant \rho \leqslant 1$; all variables vary linearly over the time interval Δt_n. If these hypotheses are applied to the governing relations (19) based on PWL models for the critical sections, the following LCP in finite increments is obtained:

$$\overline{\boldsymbol{\varphi}}_n + \rho\Delta\boldsymbol{\varphi}_n = \overline{\boldsymbol{\varphi}}_n + \rho(\mathbf{N}^{\mathrm{T}}\mathbf{Q}_{\mathrm{L}}^{\mathrm{e}}\Delta\mu_n - \mathbf{B}\Delta\boldsymbol{\lambda}_n) \leqslant \mathbf{0}, \quad \Delta\boldsymbol{\lambda}_n \geqslant \mathbf{0}, \quad (\overline{\boldsymbol{\varphi}}_n + \rho\Delta\boldsymbol{\varphi}_n)^{\mathrm{T}}\Delta\boldsymbol{\lambda}_n = 0 \tag{41}$$

The above formulation for $\rho = 1$, i.e. specialized to the backward difference scheme, is seen to coincide with the stepwise holonomic formulation Eqs. (32) and (33). The further ‘‘no new yielding’’ hypothesis leading to formulation (40), see preceding remark (b), makes the midpoint instant ρ immaterial.

(d) The above circumstances noted at (a)–(c) are noteworthy consequences of the PWL approximation, which makes the gradients of the yield functions φ_r (and of plastic potential ψ_r if different from φ_r) constant and independent from ρ. It is worth remembering that the backward difference scheme is often adopted in view of its computationally favourable properties: e.g., in dynamics algorithmic stability (in the sense of contractivity of disturbances along the step sequence) turns out to be unconditional for stable constitutive models, conditional (i.e. for Δt_n below a suitable softening-related threshold) in the presence of material instability, see e.g. Comi et al. (1992).

(e) So far in this Section the set of relationships governing the overall behaviour of the modelled frame was considered in its compact formulation (19)–(21) which permits to cast in the LCP format several variants of time-stepping analysis. However, parallel developments, omitted here for brevity, might start from the less compact equivalent formulation (15)–(18) and would lead to ‘‘mixed’’ LCPs, i.e. to problems containing both variables which are complementary (sign-constrained and orthogonal vectors) and variables which are not so, see e.g. Cottle et al. (1992). For instance, the mixed LCP which is alternative to LCP (40) and consistent with Eqs. (15)–(18), reads:

$$\Delta\boldsymbol{\varphi}_n = \mathbf{N}^{\mathrm{T}}\mathbf{E}\mathbf{C}\Delta\mathbf{u}_n - \mathbf{A}\Delta\boldsymbol{\lambda}_n \leqslant \mathbf{0}, \quad \Delta\boldsymbol{\lambda}_n \geqslant \mathbf{0}, \quad \Delta\boldsymbol{\varphi}_n^{\mathrm{T}}\Delta\boldsymbol{\lambda}_n = 0 \tag{42}$$

$$\hat{\mathbf{S}}\Delta\mathbf{u}_n - \mathbf{C}^{\mathrm{T}}\mathbf{E}\mathbf{N}\Delta\boldsymbol{\lambda}_n = \mathbf{F}_{\mathrm{L}}\Delta\mu_n \tag{43}$$

An equivalent (generally nonconvex) QP problem can be derived from the above mixed LCP simply by minimizing $-\Delta\boldsymbol{\varphi}^{\mathrm{T}}\Delta\boldsymbol{\lambda}$ to zero with respect to $\Delta\mathbf{u}_n$ and $\Delta\boldsymbol{\lambda}_n$ constrained by the inequalities Eqs. (42a) and (42b) and by Eq. (43).

(f) When Drucker's postulate holds for the PWL model of critical sections (no softening), what was stated in Section 4 for rates can be formally repeated here for finite step problems: matrices **H** and **B** become positive (semi)definite and, hence, the LCPs of this Section can be interpreted as Kuhn–Tucker conditions (necessary and sufficient for optimality) of a pair of dual convex QP problems. The consequences are important and beneficial in theoretical terms (extremum theorems in pairs) and computationally (applicability of traditional algorithms).

6. Generalized limit analysis with deformation control as a nonconvex, nonsmooth optimization

In many engineering situations classical limit analysis, based on rigid-perfectly-plastic constitutive relations, still represents an effective methodology apt to provide essential information to structural design purposes. Computationally, it requires the solution of an inequality-constrained optimization problem, specifically: a convex nonlinear mathematical programming or, for the PWL constitutive models, a LP problem (see e.g. Cohn and Maier, 1979; Maier et al., 2000; Jiràsek and Bažant, 2002).

In the present context and notation, assuming $\mathbf{H} = \mathbf{0}$ in Eq. (10a), the following couple of LP problems is generated by the statical and kinematical approach to limit analysis, respectively, leading to the safety factor s_{LA} with respect to plastic collapse:

$$s_{\mathrm{LA}} = \max_{\mu,\hat{\mathbf{Q}}}\{\mu|\hat{\mathbf{C}}^{\mathrm{T}}\hat{\mathbf{Q}} = \mathbf{F}_{\mathrm{L}}\mu + \mathbf{F}_{\mathrm{D}}, \quad \mathbf{N}^{\mathrm{T}}\hat{\mathbf{Q}} \leqslant \mathbf{Y}\} \tag{44}$$

$$s_{\mathrm{LA}} = \min_{\mathbf{u},\boldsymbol{\lambda}}\{\mathbf{Y}^{\mathrm{T}}\boldsymbol{\lambda} - \mathbf{F}_{\mathrm{D}}^{\mathrm{T}}\mathbf{u}|\mathbf{F}_{\mathrm{L}}^{\mathrm{T}}\mathbf{u} = 1, \quad \mathbf{N}\boldsymbol{\lambda} = \mathbf{C}\mathbf{u}, \quad \boldsymbol{\lambda} \geqslant \mathbf{0}\} \tag{45}$$

The (not necessarily unique) optimal vector of LP (44) and that of LP (45) define a stress state at collapse and a collapse mechanism, respectively.

The LP problems (44) and (45) could easily be shown to be dual in the sense of mathematical optimization theory (see e.g. Cohn and Maier, 1979; Gao, 1999).

The basic assumptions of traditional limit analysis may become unacceptable in a number of real-life technical problems. The main weaknesses are as follows: softening and nonassociativity are ruled out by constitutive stability in Drucker sense as underlying hypothesis; unconservative assessments of safety margins may arise due to neglected limitations in constitutive ductility and/or geometric effects of deformations on equilibrium, see e.g. Kaliszky (1996).

Conventional step-by-step methods of inelastic analysis, at present implemented in most commercial finite element computer codes, may avoid the above disadvantages of limit analysis. However, step-by-step analyses provide more information than needed in most real-life engineering situations and are in general computationally more laborious than limit analysis, especially when repeated parametric studies are required by preliminary structural design. In the presence of physical (softening) and geometrical instabilizing effects, time-marching procedures require special provisions, such as step reduction, to guarantee contractivity of disturbances in time.

Suppose that the frame response to a proportional loading process can be realistically interpreted as fully holonomic, namely as governed by Eqs. (39), where the load factor μ is nondecreasing in time, $\mu(t)$ with $\dot{\mu} \geqslant 0$, so that Eqs. (39) becomes a parametric LCP.

The mechanical circumstances which may legitimate the above assumption are basically the inexistence or negligibility of local unloading, as discussed in Section 5. Under this assumption, which reduces the plastic model to a nonlinear-elastic one, consider now the following problems:

$$s = \max_{\mu,\boldsymbol{\lambda},\boldsymbol{\varphi}}\{\mu\}, \quad \text{subject to:} \ \mathbf{L}^{\mathrm{p}}\boldsymbol{\lambda} + \mathbf{L}^{\mathrm{e}}\mu \leqslant \mathbf{M} \tag{46}$$

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}(\mathbf{Q}^{\mathrm{e}}_{\mathrm{D}} + \mathbf{Q}^{\mathrm{e}}_{\mathrm{L}}\mu) - \mathbf{B}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0}, \quad \boldsymbol{\lambda} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda} = 0 \tag{47}$$

$$s = \max_{\mu,\boldsymbol{\lambda},\boldsymbol{\varphi},\mathbf{u}}\{\mu\}, \quad \text{subject to:} \ \mathbf{L}^{\mathrm{p}}\boldsymbol{\lambda} + \mathbf{L}^{\mathrm{e}}\mu \leqslant \mathbf{M} \tag{48}$$

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\mathbf{E}\mathbf{C}\mathbf{u} - \mathbf{A}\boldsymbol{\lambda} - \mathbf{Y} \leqslant \mathbf{0}, \quad \boldsymbol{\lambda} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda} = 0 \tag{49}$$

$$\hat{\mathbf{S}}\mathbf{u} - \mathbf{C}^{\mathrm{T}}\mathbf{E}\mathbf{N}\boldsymbol{\lambda} = \mathbf{F}_{\mathrm{L}}\mu + \mathbf{F}_{\mathrm{D}} \tag{50}$$

Clearly, the former problem is the equivalent condensation of the latter through the matrix Eqs. (20) and (21). Obviously, in both problems $\boldsymbol{\varphi}$ may be eliminated through its expressions in terms of $\mathbf{u}$ and $\boldsymbol{\lambda}$, with possible (but not certain in general) computational savings.

From a mechanical standpoint, meaning and implications of the above problems are elucidated by the remarks which follow.

(a) The plastic multipliers $\boldsymbol{\lambda}$, through Eq. (11) with $\mathbf{E}^{-1} = \mathbf{0}$, govern the plastic generalized strains $\mathbf{p}$ in critical sections and, hence, through a linear relation defined by some matrix $\mathbf{L}^{\mathrm{p}}$, every deformation additional to the linear elastic one (say $\mathbf{L}^{\mathrm{e}}\mu$) due to live load. Therefore, if vector $\mathbf{M}$ defines suitable maximum admissible thresholds on such deformations, the linear inequality in Eqs. (46) and (48) enforce compliance with ductility limitations of sectional behaviour or/and serviceability requirements.
(b) At difference from static limit analysis (LA), Eq. (44), besides equilibrium and yield inequalities, also the other constitutive relationships and compatibility are enforced. Therefore, deformations are "controlled" in the sense that every feasible variable vector (i.e. obeying all optimization constraints) completely defines the/a statical situation under the load at amplification μ, account taken of hardening and/or softening behaviour of plastic hinges ($\mathbf{H} \neq \mathbf{0}$). The maximum load factor s represents the safety factor with respect to any undesirable structural event: not only plastic collapse, but also local fracture or other occurrences related to excessive deformations. Hence, the above two maximization problems avoid all the noticed limitations of classical LA. They can be interpreted as generalizations of LA, since they reduce to LP Eq. (44) for $\mathbf{H} = \mathbf{0}$, $\mathbf{M} \to \infty$ and by removing compatibility among the constraints (so that the equilibrium equations show up explicitly).

From a mathematical standpoint, the present maximization of the load factor μ, Eqs. (46)–(47) and Eqs. (48)–(50) (or minimization of $-\mu$) are nonconvex and nonsmooth nonlinear programming problems and, more specifically, optimizations under complementarity constraints. These special problems, under the denomination of MPEC, at present are the subject of intensive research in applied mathematics, especially in view of their importance for economical and management models, see e.g. Luo et al. (1996).

7. Computational aspects

In view of the multiplicity of problems formulated in what precedes within the unifying frameworks of plastic hinge frame analysis and of mathematical programming, only orientative remarks and hints to the available literature are presented in this Section on nontraditional techniques for quantitative solution of these problems.

(a) The LCP is the mathematical construct which consists of two linearly related, orthogonal, sign-constrained unknown vectors and shows up in the rate relationships throughout plasticity theory. In this paper it was shown to play a much wider, central, far reaching role as a consequence of constitutive piece-wise-linearization (PWL): in fact, "exact" and stepwise holonomic analyses have been seen to be amenable to repeated solutions of LCPs (Sections 4 and 5).

In Druckerian plasticity, with symmetric positive definite matrices (and, hence, with equivalence to convex QP) classical algorithms can be employed efficiently. Typically, a robust method is the one devised in 1965 by Lemke, based on a pivotal scheme and endowed with final termination.

In the presence of softening, which was seen to generally imply LCPs with nondefinite matrices and equivalence to nonconvex QP, recourse is necessary to more general and recent algorithms such as "Path", developed since 1994 (Ralph, 1994; Dirkse and Ferris, 1995; Ferris et al., 2001) for LCPs and nonlinear complementarity problems (NLCP) as well. Basically, this algorithm generates from the NLCP or LCP a "normal equation" system (nonlinear, nonsmooth, with Euclidean projections) and solves this by an iterative procedure which generalizes the classical line-search damped Newton method for nonlinear equations. In fact, this procedure consists of three stages in each iteration: first order approximation (trivial for LCPs); path generation; damped pathsearching. Of course, the nonconvexity and nonsmoothness of the equation residual norm to minimize imply peculiar difficulties, including the need for re-initialization in order to avoid local minima. Developments of Path algorithms and relevant software seem to currently proceed fast, fostered primarily by applications in economics and management.

A computational strategy alternative to specific LCP solvers such as Path, is recourse to general nonlinear programming algorithms (such as "interior point methods" or sequential QP, see e.g. Moré and Wright (1993); Vanderbei and Shanno (1999); Potra and Wright (2000)) applied to the equivalent nonconvex QPs formulated in Sections 4 and 5. In this strategy, two peculiar features, among others, are worth noticing and exploiting to numerical solution purposes: the absolute minimum is zero; like in LP, the solution belongs to a vertex of the feasible domain (the easy proof of this fact is here omitted for brevity).

(b) Softening hinges may cause branching in the structural response, i.e. bifurcations to capture at the rate LCP level, as seen in Section 4. The enumerative, tree-search method of Judice and Mitra (1988) guarantees to generate, in a finite number of steps, all solutions of a LCP, or to evidence that no solution exists.

The following sketchy outline of this algorithm can explain its computational performance: with reference to the compact LCP in rates ($\dot{\boldsymbol{\lambda}}$ and $\dot{\boldsymbol{\varphi}}$), Eqs. (23), m being the total number of currently active yield modes in all plastic hinges, a binary tree is generated with m levels, starting from the first pair of complementary variables ($\dot{\lambda}_1$, $\dot{\varphi}_1$) and branching at each of the 2^{m-1} nodes. In the strategy which leads to the LCP solutions along all the m levels of the above tree, each move from one node to a neighbouring one means the solution of a LP problem (precisely, the linearly constrained minimization of a λ on one branch and a $-\varphi$ on the other). Each LP solution gives guidance for skipping some descending branch ("pruning") and provides criteria for nonexistence of LCP solutions and for singling out all solutions when level m is reached. The process is initialized at a vertex of the hyperpolyhedron defined in the λ-space by the linear inequalities in Eq. (23a). This vertex or "basic feasible vector" (which represents the first node of the binary tree) is computed by LP, like in "phase one" of the classical Simplex method.

The exponential growth of the computing time with problem size (i.e. with y) is the well expected consequence of the combinatorial nature of this method. Remedies might be heuristic to prune the tree and use of traditional iterative nonlinear programming algorithms in the proximity (in the "attraction basin") of a solution.

As for the possible overall instability threshold due to softening, the copositiveness of matrix $\mathbf{B}$, Eq. (31), as sufficient and necessary stability condition, can be checked by still laborious procedures, not to be discussed herein, see e.g. Cottle et al. (1992). However, its semidefiniteness as a sufficient only condition can be assessed by consolidated techniques of linear algebra.

(c) The numerical solution of LP (44), or of (45), each one providing the solution to the other as well, can be achieved by traditional finite-termination algorithms stemming from the Simplex method devised by Dantzig and coworkers in the Forties. Alternative asymptotic iterative "interior point" algorithms have been developed in the last few years through extensive research effort and are now available to effectively solve large size problems, see e.g. Moré and Wright (1993).

(d) In the particular context of perfect plasticity ($\mathbf{H} = \mathbf{0}$), the MPEC problem of limit and deformation analysis combined was solved in Maier et al. (1979), by means of a Simplex algorithm for LP enhanced through a "restricted basis" provision, namely: the maximization of the load factor μ is carried out (subject to the linear constraints only) by pivotal transformations like for LP, but by enforcing the complementarity condition through the restriction that never both corresponding variables may simultaneously be basic at a positive level. This restriction is merely an additional rule to be obeyed in selecting the pivot in each pivotal step of the LP solution process and, hence, can be easily accommodated in the Simplex procedure (or in any of its improved versions) with finite termination at no further computational cost. However, it was empirically found that this procedure may fail when the matrix of the LCP is indefinite.

In the more general context of this paper ($\mathbf{H} \neq \mathbf{0}$ and softening), recourse is made to solution methods resulting from recent researches in mathematical programming, Luo et al. (1996) and Liu and Zhang (2002).

The main source of computational difficulties in MPEC is represented by the nonlinear, nonsmooth constraint $\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda} = 0$. Two "smoothing provisions" were proposed, based on substitution of this constraint: (i) by $\boldsymbol{\varphi}^{\mathrm{T}}\,\boldsymbol{\lambda} \geqslant -\alpha$, where α is a suitably chosen tolerance, in Ferris and Tin-Loi (2001); (ii) by $f_{\alpha}\,(\boldsymbol{\varphi}, \boldsymbol{\lambda}) = 0$, f_{α} being a smooth function depending on tolerance α, in Facchinei et al. (1999). In both cases complementarity is increasingly satisfied as $\alpha \to 0$. After this preliminary stage (which turns out to be unnecessary for small size problems like the one in the next Section), a popular robust nonlinear programming algorithm (like e.g. Sequential Quadratic Programming) might be employed repeatedly for decreasing α until convergence test is satisfied.

Clearly, penalization of the dot product $\boldsymbol{\varphi}^{\mathrm{T}}\,\boldsymbol{\lambda}$ transferred from the constraints to the objective function represents another methodological option to consider in large size problems. A MPEC arisen from the minimum cost design of offshore pipelines (Giannessi et al., 1982) was solved by making use of binary variables and branch-and-bound strategy, also apt to generate bounds bracketing the sought maximum. Unfortunately, exponential growth of computing time is expected in view of the combinatorial nature of the procedure.

(e) Three-dimensional frames may require to account for interactions of several generalized stresses in critical sections (e.g. two bending moments, axial force and torsional moment). The generation of a PWL model in more that two dimensional space may be made computationally simpler by recourse to approaches recently proposed and successfully employed for materials (Anderheggen et al., 2000; Bolzon et al., 2002; Ben-Tal and Nemirovski, 2001). When in a PWL model the (linear) yield functions are defined and matrix $\mathbf{H}$ is qualitatively selected, the hardening/softening parameters can be identified on the basis of experimental data and/or data generated by means of the original nonPWL model (cf. Bolzon et al., 2002).

As for the large number of yield modes usually implied by the PWL modelling of multidimensional plastic hinges, significant computational savings can be achieved by the following "sifting" technique: (a) in view of the assumed load proportionality, a single linear elastic analysis provides estimates $\tilde{\varphi}_i$ of the ith yield functions; (b) if $\tilde{\varphi}_i$ is below a suitable (negative) threshold, the ith mode is removed from a trial computation; (c) this mode is re-inserted in an iterated computation if it turns out to be activated despite the opposite conjecture in (b).

8. Representative special cases and numerical tests

8.1. Representative plastic hinge models

Fig. 3a shows a typical PWL relationship between bending moment Q and rotation q (a single component: $c = 1$). It involves $y = 6$ yielding modes (three for positive and three for negative bending), defined by the following six parameters (i = 1,2): ultimate moment M_{ui}; rotation at the end of the perfectly plastic stage ("breakpoint") ϑ_{bi}; "critical" rotation ϑ_{ci} at vanishing moment.

The interactions between yield modes (to be quantified by means of off-diagonal entries in hardening/softening matrix **H**, Section 2) are here limited to shrinking of flexural strength in the third quadrant (visualized by dashed lines) due to plastic deformations in the first one, and viceversa.

The analytical description of the above PWL plastic hinge model in the general format of Section 2, Eq. (2), (and with the symbols introduced there) reads:

$$\mathbf{H} = \begin{bmatrix} 0 & h_1 & -h_1 & 0 & \alpha h_1 & -\alpha h_1 \\ h_1 & -h_1 & 0 & \gamma h_1 & -\gamma h_1 & 0 \\ 0 & h_1 & -h_1 & 0 & \alpha h_1 & -\alpha h_1 \\ 0 & \beta h_2 & -\beta h_2 & 0 & h_2 & -h_2 \\ \delta h_2 & -\delta h_2 & 0 & h_2 & -h_2 & 0 \\ 0 & \beta h_2 & -\beta h_2 & 0 & h_2 & -h_2 \end{bmatrix}, \quad \mathbf{Y} = \begin{Bmatrix} M_{u1} \\ -h_1\vartheta_{b1} \\ M_{u1} \\ -M_{u2} \\ h_2\vartheta_{b2} \\ -M_{u2} \end{Bmatrix} \tag{51}$$

$$\dot{q} = \mathbf{N}\dot{\boldsymbol{\lambda}}, \quad \text{where } \mathbf{N} = \{1, 0, 0, -1, 0, 0\} \tag{52}$$

The hardening/softening matrix **H** contains six parameters, to identify through experimental data and inverse analysis. Parameters h_1 and h_2 (both negative) can be interpreted as measures of the slope of the relevant softening branches: $h_i = M_{ui}/(\vartheta_{bi} - \vartheta_{ci})$, $i = 1, 2$. Parameters α,β,γ and δ govern the assumed coupling between sagging and hugging flexural deformations.

For an easy understanding of the behaviour described by the above model, let us consider a monotonic growth of q from zero (so that $\dot{\lambda}_4 = \dot{\lambda}_5 = \dot{\lambda}_6 = 0$) and the first three yield functions, namely:

$$\varphi_1 = Q - h_1\lambda_2 + h_1\lambda_3 - M_{u1} \tag{53}$$

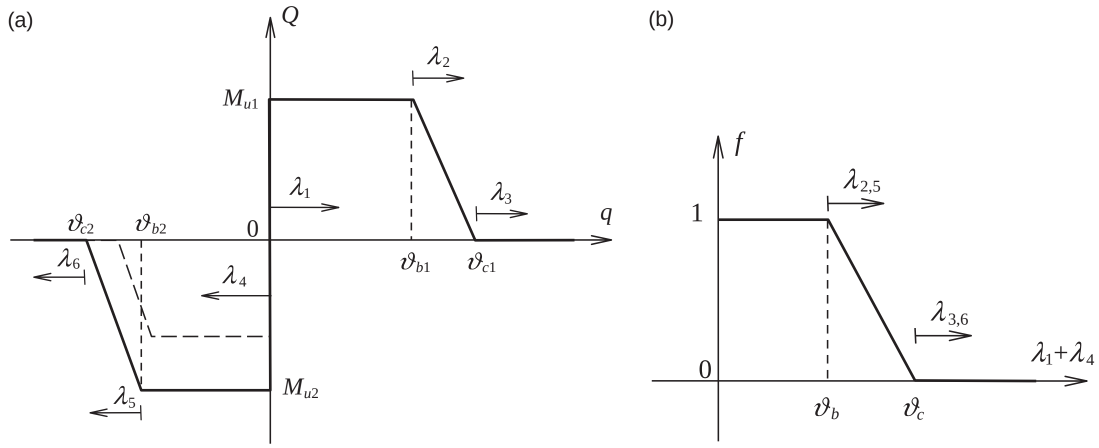

Fig. 3. (a) Plastic hinge PWL model (bending moment versus rotation). (b) The function f (defined by Eq. (59)) which governs the dependence of the peak moment on the yield processes.

$$\varphi_2 = -h_1\lambda_1 + h_1\lambda_2 + h_1\vartheta_{b1} \tag{54}$$

$$\varphi_3 = -h_1\lambda_2 + h_1\lambda_3 - M_{u1} \tag{55}$$

First, when $0 < q < \vartheta_{b1}$ and, hence, $\varphi_1 = Q - M_{u1} = 0$, $\varphi_2 < 0$ and $\varphi_3 < 0$ (perfectly plastic stage), plastic multiplier $\lambda_1 (= q)$ increases until $\lambda_1 = \vartheta_{b1}$ and $\varphi_2 = 0$.

Subsequently, both modes 1 and 2 are active (while $\varphi_3 < 0$) and, hence: Eq. (54), with $\varphi_2 = 0$ and $\lambda_1 = q$, implies that $\lambda_2 = q - \vartheta_{b1}$; Eq. (53) with $\varphi_1 = 0$, $\lambda_3 = 0$ and $\lambda_1 = q$ requires $Q = M_{u1} + h_1\lambda_2$, namely the decay of flexural strength shown in Fig. 3a for $\vartheta_{b1} < q < \vartheta_{c1}$.

Finally, for $q > \vartheta_{c1}$ all three yield modes for sagging moment are active and, hence: from Eq. (55), $-h_1(\lambda_2 - \lambda_3) = M_{u1}$; whence $Q = 0$ from Eq. (53).

The assumed particular coupling between positive and negative bending, is defined by parameters β and δ through the yield functions (with $\lambda_4 = \lambda_5 = \lambda_6 = 0$ along the considered q path):

$$\varphi_4 = -Q - \beta h_2\lambda_2 + \beta h_2\lambda_3 + M_{u2} \tag{56}$$

$$\varphi_5 = -\delta h_2\lambda_1 + \delta h_2\lambda_2 - h_2\vartheta_{b2} \tag{57}$$

$$\varphi_6 = -\beta h_2\lambda_2 + \beta h_2\lambda_3 + M_{u2} \tag{58}$$

Eq. (56) evidences the damage in terms of reduction (from $Y_4 = -M_{u2}$) of hogging flexural strength due to sagging plastic rotation $q > 0$: parameters β and δ are seen to control this effect, visualized by the dashed lines in Fig. 3a (unrealistically, no such damage for $\beta = \delta = 0$).

Clearly, parameters α and γ play for $\dot{q} < 0$ the same role as β and δ for $\dot{q} > 0$. A more general and versatile representation of coupling among yield modes may be achieved by placing further parameters instead of zeros in matrix $\mathbf{H}$, Eq. (51a).

Consider the following specialization: $M_{u1} = -M_{u2} = M_u$, $\vartheta_{b1} = -\vartheta_{b2} = \vartheta_b$, $\vartheta_{c1} = -\vartheta_{c2} = \vartheta_c$, $h_1 = h_2 = h$, $\alpha = \beta = \gamma = \delta = 1$. These constraints which reduce to 3 (from 10) the available parameters in the plastic hinge model Eqs. (51) and (52), induce a symmetry between positive and negative bending (clearly unsuitable e.g. for usual reinforced concrete beams). In this case, yield functions φ_2 and φ_3 become identical to φ_5 and φ_6, respectively, (hence, are denoted by $\varphi_{2,5}$ and $\varphi_{3,6}$) and the sums $\lambda_2 + \lambda_5 \equiv \lambda_{2,5}$ and $\lambda_3 + \lambda_6 \equiv \lambda_{3,6}$ operate the damage (instead of the λ_i individually). Therefore, the model can be given the following simplified formulation (to be used in Section 8.2):

$$\begin{cases} \varphi_1 = Q - f(\boldsymbol{\lambda})M_u \leqslant 0 \\ \varphi_4 = -Q - f(\boldsymbol{\lambda})M_u \leqslant 0 \\ \varphi_{2,5} = -h\lambda_1 - h\lambda_4 + h\lambda_{2,5} + h\vartheta_b \leqslant 0 \\ \varphi_{3,6} = -f(\boldsymbol{\lambda})M_u \leqslant 0 \end{cases}, \quad \text{where}: f(\boldsymbol{\lambda}) = 1 - \frac{\lambda_{2,5} - \lambda_{3,6}}{\vartheta_c - \vartheta_b} \tag{59}$$

$$\mathbf{H} = h\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{Y} = \begin{Bmatrix} M_u \\ M_u \\ -h\vartheta_b \\ M_u \end{Bmatrix} \tag{60}$$

$$\dot{q} = \mathbf{N}\dot{\boldsymbol{\lambda}}, \quad \text{where } \mathbf{N} = \{1, -1, 0, 0\} \tag{61}$$

The graph of Fig. 3b illustrates the behaviour of function $f(\boldsymbol{\lambda})$ involved in Eq. (59).

The PWL plastic-hinge model schematically illustrated in Fig. 2 is characterized by interaction between axial force Q_1 and bending moment $Q_2 (c = 2)$, with $y = 10$ yield modes. Of these modes, eight correspond to the sides of the octagonal domain (each endowed with a normal unit vector $\mathbf{N}_1, \ldots, \mathbf{N}_8$, Fig. 2). Modes 9 and 10 control homothetic shrinking due to softening of the strength polygon, the latter arresting the inward motion of each side. These modes are "hidden" since they cannot be visualized in the Q-space of Fig. 2, where dashed lines schematically evidence hardening (or softening) and mode interaction in the model. An analytical description of this model with homothetic shrinking can be expressed as follows:

$$\varphi_i = \mathbf{N}_i^{\mathrm{T}} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} - g(\boldsymbol{\lambda}) Y_i \leqslant 0 \quad i = 1, \ldots, 8 \tag{62}$$

$$\varphi_9 = \sum_{i=1}^{8} w_i \lambda_i - \lambda_9 - 1 \leqslant 0, \quad \varphi_{10} = -g(\boldsymbol{\lambda}) \leqslant 0 \tag{63}$$

having set:

$$g(\boldsymbol{\lambda}) = 1 - \frac{\lambda_9 - \lambda_{10}}{\omega - 1} \tag{64}$$

In inequalities (62)–(64), there are 25 available parameters: 16, namely Y_i and $\mathbf{N}_i (i = 1, \ldots, 8)$, define the onset of softening; 9, namely ω and "weights" w_i, are related to the critical kinematic quantities (like ϑ_{ci} in Fig. 3a).

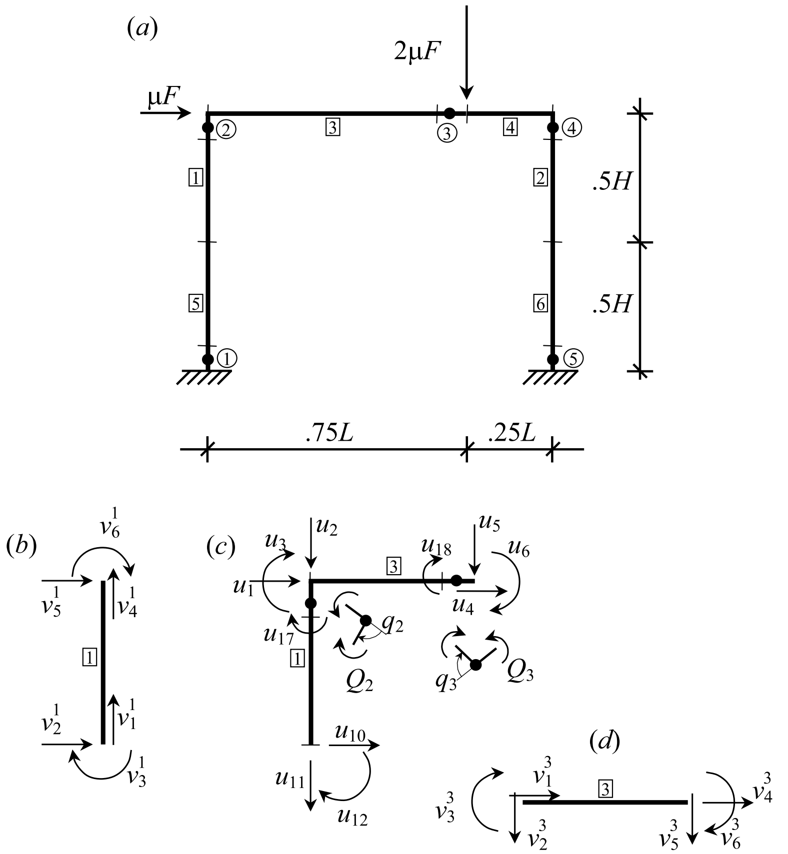

Fig. 4. Plane frame for illustrative examples: (a) geometry and loads; (b)–(d) space discretization by plastic hinges and elastic beam elements.

8.2. Illustrative example

The plane frame considered in order to test the analysis methods dealt with in what precedes is depicted in Fig. 4 as for geometry ($H = 3$ m; $L = 4$ m), reference loading ($F = 100$ kN) and FE modelling. The adopted space discretization is as follows: $n = 5$ critical sections; six elastic, cubic interpolation FEs, the first and the third of which are shown in Fig. 4b and d, each one with six nodal displacements and rotations ($v_1, \ldots, v_6$) from which three "natural" generalized strains can be deduced, namely: elongation $q_1 = v_4 - v_1$; end rotations with respect to the axis $q_2 = v_3 - (v_5 - v_2)/L$; $q_3 = v_6 - (v_5 - v_2)/L$. Fig. 4c illustrates the assemblage of beam FE and critical sections and the governing d.o.f. $u_1, \ldots, u_{20}$.

The adopted model for critical sections is described by Fig. 3 with the parameters: $M_{u1} = -M_{u2} = 236.8$ kN m, $\vartheta_{b1} = -\vartheta_{b2} = 1.362 \times 10^{-3}$ rad, for Sections 1 and 5 at the column bottoms and 3 for the beam, Fig. 4a; $M_{u1} = -M_{u2} = 420.7$ kN m, $\vartheta_{b1} = -\vartheta_{b2} = 9.103 \times 10^{-4}$ rad, for the column top Sections 2 and 4. For all

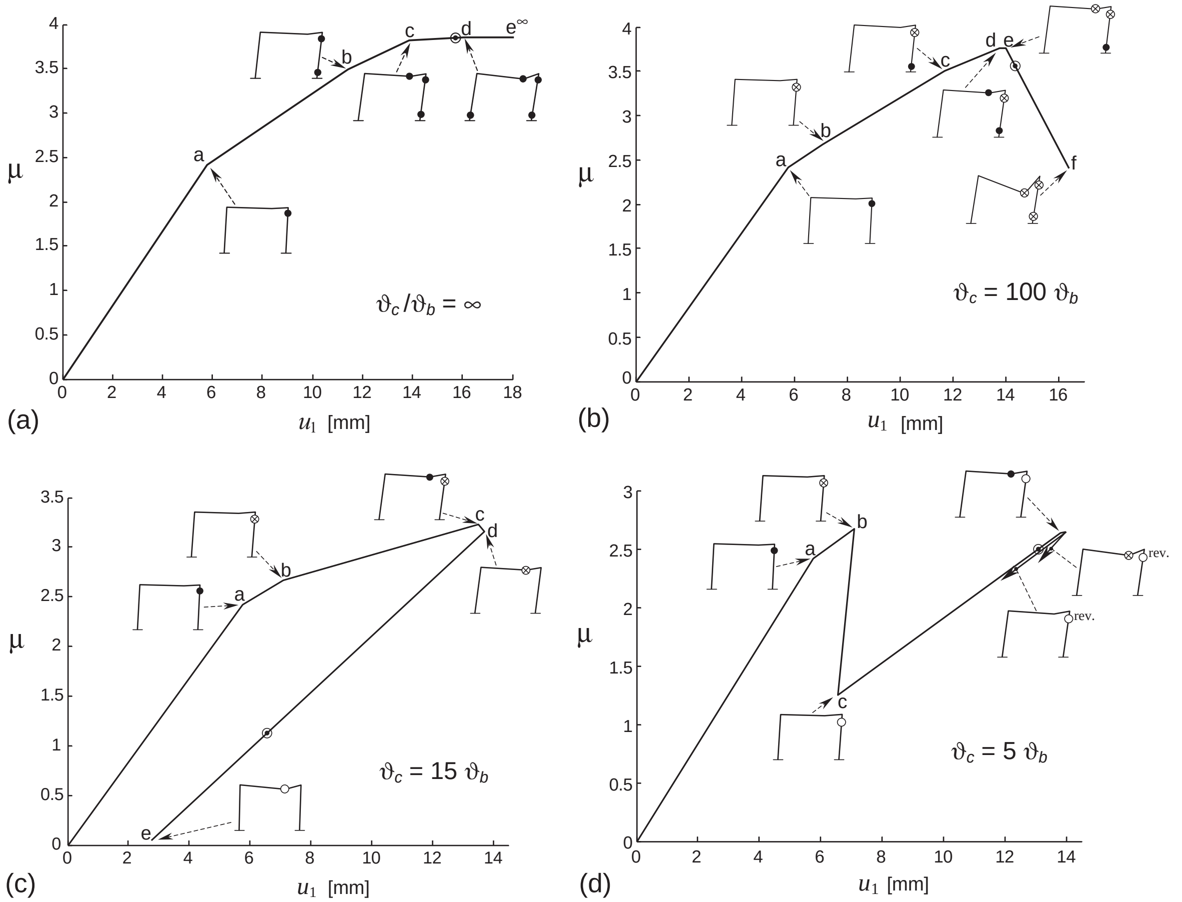

Fig. 5. Responses (in terms of horizontal displacement u_1, see Fig. 4c) to proportionally varying loads, of the frame described in Fig. 4 and endowed with the plastic hinge model of Fig. 3a, for diverse slopes of the softening branch, namely for diverse ratios (∞ i.e. perfect plasticity, 100, 15, 5) between critical, ϑ_c, and breaking point, ϑ_b, rotation. Displacements in deformed configurations are represented in a scale 30 times larger than that for frame lengths. The marks ●, ⊗, ○ denote activation of yield modes in the model of Fig. 3a, namely: perfect plasticity (● for $\dot{\lambda}_1 > 0$ or $\dot{\lambda}_4 > 0$) softening (⊗ for $\dot{\lambda}_2 > 0$ or $\dot{\lambda}_5 > 0$), free rotation (○ for $\dot{\lambda}_3 > 0$ or $\dot{\lambda}_6 > 0$); "rev." means reversed rotation). Circled dots mark the attainment of failure rotation $\vartheta_u = 10\vartheta_b$.

sections the rotation $\vartheta_{ui} = 10\,\vartheta_{bi}$ is assumed as ultimate rotation corresponding to local failure. The elastic axial and flexural stiffnesses are: 1953 MN, 33.99 MN m^2, respectively, in upper column FEs 1 and 2; 2740 MN, 79.31 MN m^2, respectively, in the other FEs (3–6).

The carrying capacity of the above portal frame in perfect plasticity ($\mathbf{H} = \mathbf{0}$) is defined by the load factor $s_{LA} = 3.857$ according to (perfectly plastic) limit analysis, Eqs. (44) or (45). This value results also from "exact" analysis (Section 4) by setting $\vartheta_c/\vartheta_b = \infty$.

The "exact" step-by-step analysis in the sense of Section 4 leads to the responses visualized in Fig. 5, for increasing softening characterized by the ratios $\vartheta_c/\vartheta_b = 100$, 15, 5 (by taking this ratio to infinity, the traditional perfect plastic hinge model is recovered). The deformed configurations at some meaningful stages are schematically shown. For $\vartheta_c/\vartheta_b = 5$, Fig. 5d, a bifurcation is found to occur at the (second peak) load $\tilde{\mu} = 2.644$ with $u_1 = 13.98$ mm and with zero moment in hinge 4: starting from this (peak) situation, for decreasing loads ($\dot{\mu} < 0$), there are two solutions (one with plastic rotations in hinges 3 and 4, another in hinge 4 only); no solution for $\dot{\mu} > 0$.

Fig. 6 visualizes the results of a parametric study carried out for varying softening branches (Fig. 3), namely by assuming $\vartheta_c/\vartheta_b = 5$, 15, 100 and $\vartheta_c/\vartheta_b = \infty$ (limit analysis) for all plastic hinges. The corresponding peak load factors turn out to be $\mu_M = 2.668$, 3.225, 3.764 and 3.857, respectively, and are indicated by stars.

The present numerical results have been obtained by "exact" integration using for the LCP solutions the enumerative method (Judice and Mitra, 1988) mentioned in Section 7, in view of expected bifurcations and the small size of the problems.

The very same peak load factors μ_M have been found through the combined limit and deformative analysis (Section 6) in its condensed formulation, Eqs. (46) and (47). The relevant MPEC problem was numerically solved by sequential quadratic programming, without smoothing provisions in view of the small number of unknowns (20 pairs of complementary variables, λ and φ, besides factor μ).

The stability tests, formulated in Section 4.3 as optional part of the exact time integration method, if applied to the present example give rise to the following remarks.

(a) Although the hardening/softening matrices $\mathbf{H}$, Eq. (60a), are not symmetric, their submatrices involved in the LCP in rates, Eqs. (23), are symmetric;

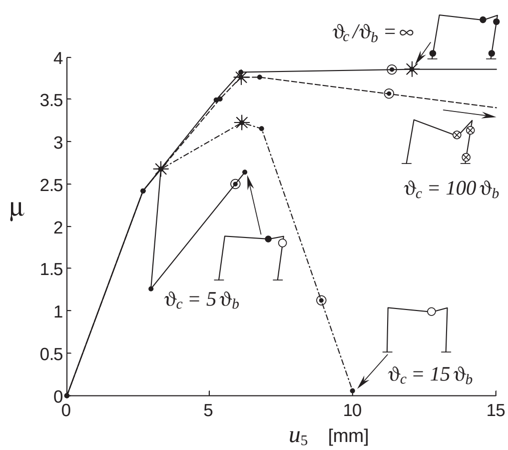

Fig. 6. Load factor μ versus vertical displacement u_5 in the frame of Fig. 4 with hinge model of Fig. 3a, for all the ratios ϑ_c/ϑ_b considered in Fig. 5. Load peaks are marked by asterisks. As for the other marks, their meanings are the same as in Fig. 5.

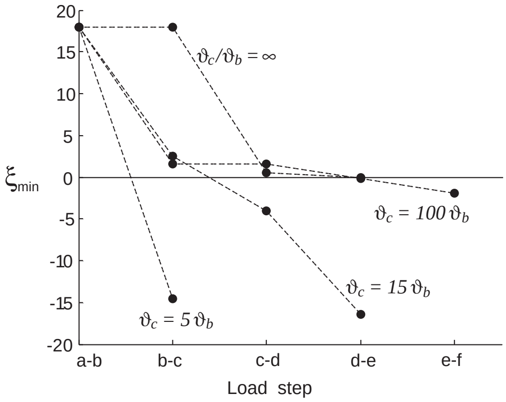

Fig. 7. Minimum eigenvalues ($\xi_{\min}$) of matrices $\mathbf{B}'$ versus the steps along the exact analyses of the frame described by Fig. 4 with the model of Fig. 3a, for the four cases of softening ϑ_c/ϑ_b considered in Figs. 5 and 6. The steps in abscissae are marked by the same letters as in Fig. 5.

(b) The sufficient and necessary stability condition Eq. (31) becomes only sufficient without sign constraints Eq. (31b) and, in view of (*a*), can be used by merely computing the minimum (real) eigenvalue, say $\xi_{\min}$, of matrices $\mathbf{B}'$, see Eqs. (23) and (31). Fig. 7 visualizes the degradation of eigenvalues ξ_{min} versus the steps along the exact step-by-step LCP analyses of the frame described by Figs. 3 and 4, for the four cases of softening ϑ_c/ϑ_b already considered in Figs. 5 and 6.

9. Closing remarks

The results presented in this paper can be summarized as follows.

(a) PWL rigid-plastic and elastic–plastic mathematical models can be formulated in terms of LCP and are apt to describe in generalized variables (bending moment, axial force ...) the plastic hinge hardening/softening behaviour in critical sections of beams and frames.
(b) Quasi-static overall analyses of frames modelled as an aggregate of linear-elastic beam finite elements and PWL critical sections, turn out to be centered on LCP or equivalent QP problems, which generally become nonconvex in the presence of softening behaviours in the hinges. The overall frame analyses include: ‘‘exact’’ time-stepping computation as a sequence of LCPs in rates; stepwise holonomic analysis with preselected loading steps; fully holonomic, single step analyses.
(c) Possible bifurcations and instability thresholds due to softening can be captured by special computational provisions concerning LCP in rates, specifically enumerative methods and copositiveness tests on a matrix.
(d) The evaluation of the safety factor with respect to, as alternatives, either plastic collapse or load peak or excessive deformations (local fracture, unserviceability) can be formulated and numerically solved as a maximization of the load factor under linear complementarity constraints, i.e. as a special case of MPEC. Such formulation can be interpreted as a generalization of the static approach of classical limit analysis by LP, with drastic reduction of the limitations intrinsic in it.
(e) Each structural analysis problem discussed in the paper in view of its numerical solution, has been related to ad hoc algorithms recently developed in mathematical programming. This is a growing area of applied mathematics which here turns out to play a unifying and beneficial role in the present and related areas of engineering mechanics of structures.

Besides the conceptual and computational advantages pointed out in this paper, PWL models entail two disadvantages: (i) the computing burden and inaccuracies implicit in any PWL approximation; (ii) the large number of variables involved by the multiplicity of yield modes. Remedies have been envisaged in the paper (Section 8), but implementations and numerical tests will be dealt with elsewhere.

When the number of complementary variables becomes large, as usual in engineering situations, the general (nondefinite) LCP and MPEC formulated herein require a selective study and implementation (to be carried out elsewhere) of recently devised algorithms.

Extensions of the present results to locking behaviour at critical sections (suitable to interpret, e.g., semirigid joints in steel frames) and to second order geometrical effects (so called P-δ effects) are straightforward. Extensions to dynamics and to large configuration changes are issues of current research.

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